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How typical is General Relativity in Brans-Dicke chaotic inflation?

Mikel Susperregi[1]

Astronomy Unit, School of Mathematical Sciences,
Queen Mary & Westfield College, London E1 4NS, United Kingdom

Abstract

General Relativity is recovered from Brans-Dicke gravity in the limit of large ω. In this article we investigate theories of Brans-Dicke gravity with chaotic inflation, allowing for either a constant or variable value of ω, known as *extended* and *hyperextended* inflation respectively. The main focus of the paper is placed on the latter. The variation ω with respect to the Brans-Dicke field is based on higher-order corrections analogous to those of the dilaton field in string theory, following the simple principle that the Brans-Dicke and metric fields decouple asymptotically. The question addressed is whether a large value of ω is predominant in most regions of the universe, which would lead to the conclusion that a typical region is then governed by General Relativity. In these theories we find that it is possible to construct inflaton potentials that drive the evolution of the Brans-Dicke field to an appropriate range of values at the end of inflation, such that a large ω is indeed typical in an average region of the universe. However, in general this conclusion does not hold and it is shown that for a wide class of inflaton potentials General Relativity is not a priori a typical theory.

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[1] *m.susperregi@qmw.ac.uk*

1 Introduction

The inflationary paradigm in cosmology is well understood and it provides a suitable framework to describe the early evolution of the universe [1]. Inflation has taken many forms and undoubtedly the key to its success is its ability to adopt changes. Although initially a inequivocal prediction of inflationary models was a density parameter $\Omega = 1$, presently a suitable choice of the scalar field potential has been shown to yield open ($\Omega < 1$) [2] or even closed ($\Omega > 1$) [3] inflation. The uncertainty in the determination of Ω from various dynamical data, such as velocity flows and other large-scale data [4], enables us to produce a large class of inflationary models that are consistent with constraints from cosmic microwave background (CMB) fluctuation maps. Hopefully, in the next few years, the values of cosmological parameters such as Ω, the cosmological constant Λ, the Hubble parameter h and the initial spectral index n will be pinned down to considerable accuracy with CMB data from the forthcoming *Planck surveyor* and ground-based CMB experiments [5]. A better determination of Ω will finally narrow down the family of inflationary models, which technically translates into a better knowledge of the inflaton potential, and ultimately on the particle physics involved.

Chaotic inflation theories in particular, are successful in explaining the generation of primordial density perturbations [1, 6, 7] and they additionally generate a quantum cosmological scenario, as we will discuss below, that one can compare with other approaches in quantum cosmology [8]. The scalar field evolves from initial large values and after a brief period of inflation, a homogeneous region becomes subdivided in further regions where the inflaton field takes a wide range of values. The scalar field reaches the end of inflation in some of these regions, whereas others continue inflating and are in turn subdivided in further regions where, as before, the field takes a wide range of values. The process of inflation is *eternal* and *self-reproducing*, in the sense that a homogeneous subregion is subdivided in inflating and non-inflating regions in much the same way as regions at earlier and later stages do [9]. At any given value of the radius of the universe, inflation still takes place and the values of the fields are thus given in terms of a probability distribution $P(\sigma)$ over an ensemble of regions of the universe that is governed by the stochastic equation motion of the scalar field [10]. In this quantum cosmological scenario $P(\sigma)$ tells us the likelihood of a certain region being at a certain stage of inflation. These theories are unable to make definite predictions for any given region due to the stochastic nature of inflation. The self-reproducing scenario also tells us that although constraints on the age of the universe obtained via stellar evolution, element abundances, etc, will ultimately yield an estimate of the time elapsed since our region of the universe stopped inflating, those constraints will not give us any information of the age of the universe as a whole. The property of self-reproduction permits the possibility of a universe that extends to the infinitely remote past. On the other hand, it has been shown that, given a homogenous region where inflation sets on at an initial time, then the distribution $P(\sigma)$ quickly approaches a stationary regime [11], and is therefore solely dependent on the initial conditions.

In the case of several scalar fields, the interplay between the evolution of a scalar field and the inflaton field influences the course of inflation, and the *self-reproducing universe* is then described in terms of the joint probability distributions $P(\sigma; \Theta)$, where Θ denotes scalar fields that are coupled to inflation.[2] The joint evolution of the fields produces a quantum cosmological scenario where the fields evolve over a self-similar ensemble of regions. One such scalar field is the Brans-Dicke (BD) field Φ, first introduced in the context of more generalized theories of gravity [12].

BD gravity was initially motivated by Mach's principle and a dimensional argument due to Dennis Sciama [13] that relates the magnitude of G, the horizon radius H^{-1} and the total mass M within the horizon via $GMH \sim 1$. The BD field determines the magnitude of G (and therefore M_P) and is slowly-varying over horizon scales and its coupling coefficient to the curvature is denoted by ω. In string theory, the BD field arises for the one-loop string effective action in the form of the *dilaton* field [14], the equivalent of a BD field with variable ω. Inflationary cosmology can naturally adopt BD gravity and the result is the so-called *extended inflation* (constant ω) that was first suggested by La and Steinhardt [15]. Extended inflation revives the old inflation scenario [16] in that a first-order transition is responsible for inflation, but the addition of the BD term in the action terminates the phase transition with a bubble spectrum that is consistent with structure formation. However, limits on CMB anisotropies suggest that extended inflation can only work for $\omega \leq 25$ [17], in contrast with time-delay experiments [18] that set a lower bound $\omega \gtrsim 500$. This incompatibility was reconciled by introducing *hyperextended inflation* [19, 20], which allows for a variable ω. In spite of the apparent incompatibility of the CMB data with extended inflation, these theories have been studied in some detail, following the thesis that the self-reproducing scenario can lead to a wide range of spectra of bubble sizes, depending on the initial value of the Φ field (see last reference in [17] for a discussion of this argument to reconcile CMB data with extended inflation). The distributions $P(\sigma, \Phi)$ for several extended inflation models have been derived, the spectrum of density fluctuations at the end of inflation [21, 23] and models of formation of cosmic structure [7].

The purpose of this paper is to investigate in more detail some implications of hyperextended inflation and ask ourselves how typical General Relativity (GR) is in these theories. GR is an accurate theory of gravity in our neighbourhood, and a large value $\omega \geq 500$ reduces BD gravity to GR. We compute the probability distribution of $\omega(\Phi)$ in hyperextended inflation and address the question of whether the most typical values of ω at the end of inflation in such a universe are compatible with the large values required to recover GR. The functional dependence of ω on the BD field Φ is given by higher-order corrections in the effective string action, following the *principle of least coupling* [14]. This principle states that the field Φ will evolve in a way that in the asymptotic regime its coupling to matter will vanish. The principle of least coupling enables us therefore to have a convergent and well-behaved approximate form of ω so that we can calculate the likelihood of an arbitrary

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[2]A more coarse description would be achieved by computing the partial distribution for σ, if one is not interested in the values of the other scalar fields, by computing the integral $\int \mathrm{d}\Theta\, P(\sigma; \Theta)$.

value of ω in any region of the universe. The conclusion of our analysis is that although inflaton potentials can be chosen so that the likeliest values of ω are driven towards large values compatible with GR, in general the reverse is not true, and it is not the case that for an arbitrary potential GR is a typical theory.

This article is structured as follows. Section 2 gives a brief summary of the essentials of extended inflation. The reader familiar with this can skip this section and proceed directly to Section 3, where hyperextended inflation is studied and applied to the case of powerlaw potentials. In Section 3 we compute the probability distributions $P(\sigma, \Phi)$, volume ratios and in general we discuss the likelihood of physical quantities, in particular the probability $P(\omega)$ for some simple Ansatze of ω. In Section 4 we discuss these results and investigate arguments of naturalness and typicality to look into the question of how typical GR is in terms of $P(\omega)$. In Section 5 we sum up with some conclusions.

2   Extended inflation

Extended inflation is governed by the action [15, 21]

$$ \mathcal{S} = \int \mathrm{d}^4 x \sqrt{-g} \left[\Phi\, R - \frac{\omega}{\Phi} (\partial \Phi)^2 - \frac{1}{2} (\partial \sigma)^2 - V(\sigma) \right], \tag{1} $$

where R is the curvature scalar and $V(\sigma)$ the inflaton potential. The potential energy of the BD field is assumed to be zero or negligible in comparison to that of the inflaton field. The Planck mass is related to the BD field,

$$ M_P^2(\Phi) = 16\pi\, \Phi. \tag{2} $$

The beginning-of-inflation boundary (BoI) in (1) is given by

$$ V(\sigma) = M_P^4(\Phi), \tag{3} $$

and similarly the end-of-inflation boundary (EoI) is marked by the condition

$$ \frac{1}{2}\dot{\sigma}^2 + \omega\, \frac{\dot{\Phi}^2}{\Phi} \approx V(\sigma). \tag{4} $$

The resulting equations of motion in a FRW expanding background read, in the slow-roll approximation, i.e. $\ddot{\Phi} \ll H\,\dot{\phi} \ll H^2 \Phi$, $\dot{\sigma}^2 + 2\omega\,\dot{\Phi}^2/\Phi \ll 2\,V(\sigma)$,

$$ \frac{\dot{\Phi}}{\Phi} = 2\,\frac{H}{\omega}, \tag{5} $$

$$ \dot{\sigma} = -\frac{1}{3H}\,V'(\sigma), \tag{6} $$

$$H^2(\sigma, \Phi) = \frac{1}{6\Phi} V(\sigma). \tag{7}$$

As is shown in [21], the following conservation law follows from (5)-(7):

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\omega\Phi + \int \mathrm{d}\sigma \, \frac{V(\sigma)}{V'(\sigma)} \right] = 0. \tag{8}$$

The classical trajectories of the fields are then given by the integrals of (8). In the (σ, Φ) plane the integral of (8) is a parabola in the case of a powerlaw potential, a straight line for an exponential potential and quasi-logarithmic trajectories for double-well potentials (these trajectories are investigated in [21]). A totally classical analysis therefore enables us to write the orbits of the fields in terms of the initial conditions (σ_0, Φ_0), and the motion is restricted to one degree of freedom on the plane (σ, Φ). In addition to the classical motion, quantum diffusion is responsible for "jumps" of the fields between classical trajectories, and therefore a large number of classical trajectories are accessible after a certain period of evolution, as is predicted in the self-reproducing universe model.

In the case of a powerlaw potential $V(\sigma) = \lambda/(2n) \, \sigma^{2n}$ the curve (8) becomes

$$\Phi = \left(\Phi_0 + \frac{1}{4n\omega} \sigma_0^2 \right) - \frac{1}{4n\omega} \sigma^2, \tag{9}$$

as is shown in Fig. 1, with the corresponding BoI and EoI curves (3)(4). The EoI curve is always a parabola, whereas the BoI curve is a straight line for $n = 1$, a parabola for $n = 2$. For $n > 2$ both curves intersect at a certain value Φ_{max}, given by

$$\Phi_{\mathrm{max}} = \frac{1}{4n^2} \left(\frac{3\omega - 2}{\omega} \right)^{n/(n-2)} \left(\frac{32\pi^2}{\lambda n^3} \right)^{1/(n-2)}. \tag{10}$$

Therefore the region between the curves BoI and EoI on the (σ, Φ) where inflation takes place is bounded in the case of $n > 2$ (region enclosed between the thick solid and dotted lines in Fig. 1). Inflation will set on if the initial BD field is within the range $0 < \Phi_0 < \Phi_{\mathrm{max}}$ on the curve BoI and the period of inflation decreases steadily until it vanishes at $\Phi_0 = \Phi_{\mathrm{max}}$. The alternative case of $n \leq 2$ gives rise to the so-called 'run-away' solutions that we discuss in the following section.

The ratio of energy densities of the fields σ, Φ is

$$\frac{\rho_\Phi}{\rho_\sigma} \sim \frac{\Phi}{3\omega^2}, \tag{11}$$

and whereas the energy density of σ is dominated by the potential energy, Φ is entirely driven by its kinetic energy. From (11) we see that at large Φ the energy density of the field Φ overcomes that of inflation.

2.1 'Run-away' solutions

These solutions occur in the case of $\Phi_{\mathrm{max}} \to \infty$ ($n \leq 2$ in the case of a powerlaw potential). Even though the classical trajectory (9) is a parabola for an arbitrary large initial BD field Φ_0, quantum diffusion drives the fields to larger and larger values and, for as long as inflation takes place, a tendency towards greater values of Φ is enhanced [21]. Quantum jumps that take the fields from the initial classical trajectory to other classical trajectories corresponding to larger values of the initial fields are favoured. This process goes on indefinitely and the period of inflation is prolonged for those regions where jumps to larger values of the fields take place. Therefore, the largest physical volumes are occupied by values of the fields that grow without limit, where the period of inflation is indefinitely long. These so-called *run-away solutions* do not permit us to make predictions on the most typical values of the fields. The volume of the universe is almost in its entirety occupied by regions of arbitrarily large M_P. These solutions are only compatible with the observed universe with negligible probability and are therefore ruled out.

For $n > 2$, as seen in Fig. 1, the BoI and EoI boundaries (3)(4) intersect at a value $\Phi = \Phi_{\mathrm{max}}$, and it is that value of Φ that occupies the largest fraction of the total physical volume. These theories make a definite prediction of the likeliest Planck mass at the end of inflation

$$M_{P*}^2 = 16\pi\,\Phi_{\mathrm{max}}, \tag{12}$$

where Φ_{max} is given by (10). In the limit of large ω, λ thus satisfies the following relation

$$\lambda = 2n \left(\frac{12\pi}{n^2}\right)^n \left(\frac{1}{M_{P*}^2}\right)^{n-2}. \tag{13}$$

Therefore, if a $n > 2$ powerlaw potential is the right theory and the Planck mass (12) is given by its value in our region of the universe, $M_{P*}^2 \sim 10^{19}$ GeV, under the assumption that our immediate neighbourhood is a *typical* region, λ results in the following order of magnitude

$$\lambda \sim 10^{-17} \tag{14}$$

for $n = 3$. This is effectively an upper limit for the order of magnitude of λ predicted by (13), as it decreases sharply for greater values of n. We can also conclude from (13) that larger values of λ can only be realistic if the typical value of M_{P*}^2 is several orders of magnitude smaller than the value measured in our region of the universe.

2.2 Stationary universe

The extended inflation theory describes a self-reproducing universe where the values of the fields (σ, Φ) are described in terms of probability distributions. In this section, we will

summarize some results (see e.g. [21, 22, 23]). The comoving probability $\mathcal{P}_c(\sigma, \Phi, t)$ satisfies the convervation equation [1, 10]

$$\partial_t \mathcal{P}_c = -\partial_\sigma J_\sigma - \partial_\Phi J_\Phi, \tag{15}$$

where the probability current $\vec{J} \equiv (J_\sigma, J_\Phi)$ is given by, in the slow-roll regime where the effect of quantum diffusion can be neglected,

$$J_\sigma \approx -\frac{M_P^2(\Phi)}{4\pi} H^{\alpha-1} \partial_\sigma H \, \mathcal{P}_c, \tag{16}$$

$$J_\Phi \approx -\frac{M_P^2(\Phi)}{2\pi} H^{\alpha-1} \partial_\Phi H \, \mathcal{P}_c, \tag{17}$$

The index α denotes the choice of time parametrization. The synchronous gauge is recovered in the case of $\alpha = 0$, which corresponds to $t = \log a$, and $\alpha = 1$ corresponds to proper or cosmic time, $t = \tau$. Henceforth we will adopt the synchronous gauge. As it is apparent from the results discussed in [22], the probability distributions are very sensitive to the choice of time variable. Nonetheless they are helpful in giving a qualitative picture of the dynamics of the fields and whether they reach asymptotic values or grow indefinitely. Also we shall see that the physical probabilities (or equivalently, the fraction of the physical volume occupied by a homogeneous hypersurface (σ, Φ)=const) can be computed in a way that is insensitive to the choice of time parameter. This procedure was first suggested in [25] for an inflation-only scenario, and implemented in the case of extended inflation in [21]. In order to transform (15) into an eigenvalue equation, we consider the following expansion:

$$\mathcal{P}_c(\sigma, \Phi, t) = \sum_{n=1}^{\infty} \psi_n(\sigma, \Phi) \, e^{-\gamma_n t}, \tag{18}$$

where $\gamma_1 < \gamma_2 < \gamma_3 < ...$, and thus in the limit $t \to \infty$ the dominant contribution is that of $n = 1$, $\mathcal{P}_c \sim \psi_1 \, e^{-\gamma_1 t}$. The value of γ_1 depends on the form of the potential, and is determined by the boundary condition that establishes the conservation of probability flux along the EoI boundary. For a powerlaw potential typically we have $7.8 \lesssim \gamma_1 \lesssim 8.3$ for $10^{-18} \lesssim \lambda \lesssim 10^{-15}$ (we use these small values of λ in view of (13)). By substituting (16)(17) into (15), it is easy to calculate the following asympotic solution:

$$\mathcal{P}_c(\sigma, \Phi, t) \sim C_0 \, \Phi^{\frac{\omega}{2}\gamma_1 - 1} \left(\frac{V}{V'}\right) e^{-\gamma_1 t}, \tag{19}$$

where C_0 is a normalization constant. The comoving probability (19) gives us an idea of the likelihood of certain configurations in different regions of the universe. However, for those regions in the (σ, Φ) plane that lie between the BoI and EoI curves (3)(4), the physical volumes are many orders of magnitude greater than those regions that have undergone thermalization at the same comoving scale. Therefore, in order to address the question of relative likelihood of certain physical quantities (such as the value of the Planck mass), we need to look into the actual physical volumes of homogeneous hypersurfaces. By virtue of the principle of stationarity, the fraction of the volume of a given hypersurface with respect

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to the total volume reaches an asymptotic value, and it is therefore possible to examine the question of how typical a quantity is in terms of volume ratios.

There are two equivalent ways to tackle this problem. The first one is to write (15) in physical coordinates, for a physical probability distribution $\mathcal{P}_p$ (rather than the comoving $\mathcal{P}_c$), by adding an extra term $3H\mathcal{P}_p$ on the RHS of (15). The presence of this extra term, that accounts for the background expansion, complicates the problem and the eigenvalue equation is no longer soluble analytically.[3] An second procedure, that we will use here, is to compute the physical volumes of the hypersurfaces (σ,Φ)=const. Given a suitable normalization, the volumes of homogeneous hypersurfaces can be expressed in terms of the fraction of the total volume or the volume of the thermalized regions [25, 26]. As it is shown in the Appendix, the ratio of the physical volume $\mathcal{V}(\sigma, \Phi)$ of an arbitrary hypersurface with respect to the thermalized volume $\mathcal{V}_*$ for a powerlaw potential is given by

$$r(\sigma, \Phi) = \frac{\mathcal{V}(\sigma, \Phi)}{\mathcal{V}_*} \sim \Phi^{\frac{\omega}{2}\gamma_1}\left(1 + \frac{\sigma^2}{4n^2\,\omega^2}\right)^{1/2}. \tag{20}$$

The likelihood of a point lying on the hypersurface (σ,Φ)=const is proportional to its volume, so r gives a measure of this likelihood. Comparing (20) and (19), we notice that in both cases the same tendency is preserved, that of the field σ rolling down towards the minima of $V(\sigma)$, whereas the BD field Φ tends to increase towards the largest values possible.

3   Variable ω: hyperextended inflation

In this section we generalize the results of §2 for a dynamical ω. This theory was suggested by [19, 20], motivated by observational discrepancies of the extended inflation model pointed out in [17]. The CMB data analysis of [17] requires $\omega \le 25$, as opposed to the constraint $\omega > 500$ of [18], as was discussed in the Introduction. The hyperextended inflation action is again given by (1), where ω is dependent on Φ. The resulting equations are

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} = \frac{\omega(\Phi)}{\Phi^2}\left[\partial_\mu\Phi\,\partial_\nu\Phi - \frac{1}{2}g_{\mu\nu}\,(\partial\Phi)^2\right] + \frac{1}{\Phi}\left(\nabla_\mu\nabla_\nu\Phi - g_{\mu\nu}\,\Box\Phi\right) + \frac{8\pi}{\Phi}\,T_{\mu\nu}, \tag{21}$$

$$\Box\Phi = \frac{1}{2\omega(\Phi) + 3}\left[8\pi T^\mu_\mu - \omega'(\Phi)(\partial\Phi)^2\right], \tag{22}$$

$$\Box\sigma = -V'(\sigma), \tag{23}$$

where the energy-momentum tensor of the matter sector $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = \frac{1}{16\pi}\left[\partial_\mu\sigma\,\partial_\nu\sigma - \frac{1}{2}g_{\mu\nu}\,(\partial\sigma)^2 + g_{\mu\nu}\,V(\sigma)\right]. \tag{24}$$

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[3]A gauge-invariant approach for calculating the spectrum of density fluctuations was pursued in [23] along these lines, without an explicit derivation of $\mathcal{P}_p$.

We examine the solutions in a FRW metric:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\Big[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\Big], \tag{25}$$

where $k = 0, \pm 1$ and $a(t)$ is the scale factor of the universe. Then the field equations (21)-(23) become

$$H^2 + \frac{k}{a^2} + H\frac{\dot{\Phi}}{\Phi} = \frac{1}{6}\,\omega\,\frac{\dot{\Phi}^2}{\Phi^2} + \frac{1}{6\Phi}\Big[V(\sigma) + \frac{1}{2}\dot{\sigma}^2\Big], \tag{26}$$

$$\dot{H} + 3H^2 + 2\frac{k}{a^2} + \frac{5}{2}H\frac{\dot{\Phi}}{\Phi} = -\frac{1}{2}\frac{\ddot{\Phi}}{\Phi} + \frac{1}{2\Phi}V(\Phi), \tag{27}$$

$$\ddot{\Phi} + 3H\dot{\Phi} = \frac{1}{2\omega+3}\Big[2V(\sigma) - \frac{1}{2}\dot{\sigma}^2 - \omega'\dot{\Phi}^2\Big], \tag{28}$$

$$\ddot{\sigma} + 3H\dot{\sigma} = -V'(\sigma), \tag{29}$$

where, as is customary $H \equiv \dot{a}/a$. In the slow-roll approximation, (26)-(29) become

$$\frac{\dot{\Phi}}{\Phi} = 2\Sigma\frac{H}{\omega}, \tag{30}$$

$$\dot{\sigma} = -2\Phi\frac{V'}{V}H, \tag{31}$$

$$H^2 = \frac{V(\sigma)}{6\Phi}, \tag{32}$$

where $\Sigma \equiv (1 - \Phi\omega')$. These equations are essentially identical to (5)-(7), with the exception of the Σ factor in (30). For negligible variations of ω, such that $\omega' \ll \Phi^{-1}$, both models are naturally the same, i.e. $\Sigma = 1$. The departure of Σ from unity is the characteristic of the hyperextended model. The EoI boundary is given by

$$\frac{1}{2}\dot{\sigma}^2 + \frac{\omega}{\Sigma^2}\frac{\dot{\Phi}^2}{\Phi} = V(\sigma), \tag{33}$$

that yields, with the aid of (30)-(32),

$$\Phi_* = \left(\frac{3\omega_* - 2}{\omega_*}\right)\left(\frac{V}{V'}\right)_*^2, \tag{34}$$

where as usual $*$ denotes the values of the quantities at the end of inflation. It is easy to see that (34) is in fact formally equivalent to (4), only that ω_* is dependent on Φ_*, so (4) is solved for Φ_* once the functional dependence of ω is determined. The BoI boundary is given by (3) as in extended inflation. From (30)(31) the following conservation law follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int\frac{\omega}{\Sigma}\,\mathrm{d}\Phi + \int\frac{V}{V'}\,\mathrm{d}\sigma\right) = 0, \tag{35}$$

which is a generalization of (8). The form of the orbits given by (35) will strongly depend on the model for $\omega(\Phi)$ that we use. Qualitatively, we see from (35) that the classical orbits depart more from the extended inflation ones (8) for smaller values of Φ (corresponding to early stages of inflation) than larger ones (close to EoI boundary). In principle the hyperextended inflation action does not contain sufficient information to determine the variations of ω, and we need to make an Ansatz for its functional dependence. In the following sections we will investigate this problem.

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3.1 Principle of least coupling

In string theory the BD field Φ appears naturally in the form of the dilaton Ψ, a scalar field associated to the graviton tensor field. The effective action for the graviton-dilaton-inflation sector is given by [14]

$$\mathcal{S} = \int d^4x \sqrt{-g}\, e^{-2\Psi} \left[B_g(\Psi)\, R + B_\Psi(\Psi)\, 4(\partial \Psi)^2 - \frac{1}{2} B_\xi(\Psi)\, (\partial \sigma)^2 - V(\Psi, \sigma) \right], \qquad (36)$$

where it is assumed that the string tension $\alpha = 1$; $B_i(\Psi)$ are coupling functions. The matter couplings $B_i(\Psi)$ of the dilaton, which play a rôle equivalent to the BD coupling $\omega(\Phi)$, are responsible for deviations from GR. Little is known of the general form of $B_i(\Psi)$, which depend on the details of the perturbative higher-loop corrections of the scalar-tensor interactions. However, a simple assumption of universality of these coupling functions, such that $B_i(\Psi) = B(\Psi)$ leads to a simple and interesting model. The cosmological evolution of the graviton-dilaton-matter system under this assumption drives the dilaton to a massless regime (as is shown in [24]) and thus it decouples from matter asymptotically. This is also known as the *principle of least coupling*, that we will assume in this paper by applying it to $\omega(\Phi)$. The coupling function $B(\Psi)$ is therefore given as a Taylor expansion in the string coupling $g_s^2 \equiv e^{2\Psi}$ [24]:

$$B(\Psi) = b_0 + b_1\, e^{2\Psi} + b_2\, e^{4\Psi} + \dots. \qquad (37)$$

These expressions translate into the BD formalism in the following way. The dilaton Ψ relates to the BD field Φ via

$$\Phi \equiv \exp(-2\Psi), \qquad (38)$$

and thus $g_s^2 = \Phi^{-1}$. The BD coupling $\omega(\Phi)$ is given then by

$$\omega(\Phi) = \eta_0 + \frac{\eta_1}{\Phi} + \frac{\eta_2}{\Phi^2} + \frac{\eta_3}{\Phi^3} + \dots, \qquad (39)$$

where η_0 is given by the low-energy value predicted by string theory and the higher-order coefficients η_i are determined by string-loop corrections. Although η_0 is strongly dependent on the mechanisms of compactification and supersymmetry breaking, it is widely accepted that $\eta_0 = -1$ in four dimensions. We must rely on observational constraints or string-loop calculations to estimate the remaining η_i.

In a cosmological inflationary model, we have seen in §2 that the course of inflation leads the BD field to grow, in some cases without limit. Thus g_S^2 is a good perturbation parameter and it is apparent from (39) that in a hypothetical asymptotic case of $\Phi \to \infty$, the most probable value of ω is -1 and any other value takes place with negligible probability. A large positive value of ω can be probable only if the BD field is bounded above, i.e. $\Phi_{\max} < \infty$, and provided that the contribution of the terms $\eta_n/\Phi_{\max}^n$ in (39) is not negligible with respect to unity. We have already seen in §2 that in extended inflation it is also the case that only models with finite $\Phi_{\max}$ are realistic from the astrophysical point of view.

Let us investigate the following toy model. We assume that a large value of ω is pertinent so that it is consistent with the observational bound $\omega \gg 500$, and the inflation potential such that Φ is bounded above, e.g. a powerlaw potential $V(\sigma) = \lambda/(2n)\,\sigma^{2n}$ with $n > 2$. The coupling $\omega(\Phi)$ is truncated for simplicity to the one-parameter Ansatz

$$\omega(\Phi) = -1 + \frac{\eta}{\Phi}, \tag{40}$$

and $\eta > 0$. Inflation occurs for values of the BD field that grow from an initial value Φ_0 to a maximum value $\Phi_{\max}$ that is given by[4]

$$\Phi_{\max} = \frac{3^{n/(n-2)}}{4n^2} \left(\frac{32\pi^2}{\lambda n^3} \right)^{1/(n-2)}. \tag{41}$$

Most regions are occupied by $\Phi_{\max}$, as is shown in §2.1. In these regions $\omega_{\text{typical}} \gg 1$, and thus $\eta \approx \Phi_{\max}\,\omega_{\text{typical}}$. Therefore

$$\omega(\Phi) \approx \omega_{\text{typical}}\,\frac{\Phi_{\max}}{\Phi}. \tag{42}$$

This toy model is schematically illustrated in Fig. 2. The value of ω decreases monotonically during the course of inflation from a given arbitrary value ω_0 to its EoI inflation value, $\omega(\Phi_*)$, the most likely of which is $\omega_{\text{typical}} = \omega(\Phi_{\max})$. On the other hand, if $\omega_0 \approx -1$, such that $\eta \ll \Phi_0$, then $\omega \approx -1$ throughout.

From the conservation law (35) follows that the trajectories of the fields are given by

$$2\eta\,\log(\eta + \Phi) - \Phi + \frac{1}{4n}\,\sigma^2 = \text{const.} \tag{43}$$

These trajectories reduce to the parabolical orbits described by (8) in the approximation $O(\Phi^2/\eta^2)$, whereas the following order of the expansion of the logarithmic term in (43) leads to

$$\Phi - \frac{1}{\eta}\,\Phi^2 + \frac{1}{4n}\,\sigma^2 = \text{const.} \tag{44}$$

It is easy to see that, in comparison to (9), for sufficiently large η, the trajectories (44) lead to larger final values of Φ relative to the parabola (9) for similar initial conditions. This effect can only be reproduced in (9) by means of quantum fluctuations that take the fields from one parabola to an outer one where the fields are larger. The approximation (44) enables us to constrain η (and therefore ω_{typical}) in this simple model in terms of the initial conditions (σ_0, Φ_0), given that $\Phi_{\max}$ is fixed by the choice of potential. In qualitative terms, it can be also said that due to the growth of Φ during the course of inflation, ω is always greater at the initial time (in the model described by (40)) than at the end of inflation.

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[4] We use (10) under the approximation $(3\omega_* - 2)/\omega_* \to 3$ for large ω. Therefore the EoI boundary is $\Phi_* \approx 3\sigma_*^2/4n^2$

3.2 Probability distributions and volume ratios

We now compute the probability distributions and volume ratios of homogeneous hypersurfaces, in order to generalize the results of §2 for variable ω. In the following subsection we will apply these results to the simple one-parameter Ansatz that we have briefly discussed above. From (30)-(32) it is easy to see that the probability current (J_σ, J_Φ) is given by

$$J_\sigma = -2\Phi \left(\frac{V'}{V}\right) \mathcal{P}_c, \tag{45}$$

$$J_\Phi = 2\Phi \left(\frac{\Sigma}{\omega}\right) \mathcal{P}_c. \tag{46}$$

The continuity equation (15) then yields, in the limit $\mathcal{P}_c \sim \Psi_1 \, e^{-\gamma_1 t}$,

$$\mathcal{P}_c(\sigma, \Phi, t) = C_0 \left(\frac{V}{V'}\right) \Phi^{-1} \left(\frac{\omega}{\Sigma}\right) \exp\left(\frac{\gamma_1}{2} \int \frac{\omega}{\Phi\Sigma} \, d\Phi - \gamma_1 t\right). \tag{47}$$

Furthermore, (45)(46) enable us to compute the regularized volumes of thermalized regions and homogeneous hypersurfaces via (75)(76). Once again we focus on the particular case of a powerlaw potential. It is easy to show that the thermalized volume is, to a good approximation, insensitive to variations of ω, and therefore it is given by the extended inflation result (78), where ω is evaluated at $\omega(\Phi_{\max})$. The volume of a homogeneous hypersurface is on the other hand

$$\mathcal{V}(\sigma, \Phi)_{\text{regularized}} = 2\mathcal{V}_0 \left|\frac{e^{(3-\gamma_1)\,t_c} - 1}{3 - \gamma_1}\right| \left(\frac{\omega}{\Sigma}\right) \left(1 + \frac{\sigma^2}{4n^2\omega^2} \Sigma^2\right)^{1/2} \exp\left(\frac{\gamma_1}{2} \int \frac{\omega}{\Phi\,\Sigma} \, d\Phi - \gamma_1 t\right), \tag{48}$$

where t_c is the cut-off time as is explained in the Appendix. The volume ratio r of a homogeneous hypersurface with respect to the thermalized volumes is then

$$r(\sigma, \Phi) = \frac{\gamma_1}{2} \, \Phi_{\max}^{-\frac{\omega}{2}\gamma_1 - 1} \, \omega(\Phi_{\max}) \left(\frac{\omega}{\Sigma}\right) \exp\left(\frac{\gamma_1}{2} \int \frac{\omega}{\Phi\,\Sigma} \, d\Phi\right) \left(1 + \frac{\sigma^2}{4n^2\omega^2} \Sigma^2\right)^{1/2}. \tag{49}$$

It is straightforward also from (49) to compute the relative likelihoods of (σ, Φ) and $(\tilde{\sigma}, \tilde{\Phi})$ by working out the ratio $\tilde{r}/r$. For two homogeneous hypersurfaces that are only differentially separated, this ratio becomes

$$\frac{\tilde{r}}{r} = 1 + \frac{\sigma}{4n^2} \, \delta\sigma - \frac{2}{\Phi} \, \delta\Phi. \tag{50}$$

By comparing (49) with (20), we observe that the multiple presence of the $\Sigma(\Phi)$ factor in (49) can potentially yield a very different result, in comparison to extended inflation, depending on $\omega(\Phi)$. This factor is, from (39),

$$\Sigma(\Phi) = 1 + \frac{\eta_1}{\Phi} + 2\frac{\eta_2}{\Phi^2} + 3\frac{\eta_3}{\Phi^3} + \dots, \tag{51}$$

and therefore one recovers extended inflation, $\Sigma \approx 1$, in the limit $\Phi_{\max} \to \infty$ and also, as Φ increases throughout the course of inflation, the predictions of (49) differ less from those

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of (20) the closer we get to the EoI boundary, and they differ most at the early stages of inflation. In the most general case (39), the ratio (51) requires a numerical resolution, but we can see that the result is roughly of the form

$$r(\sigma, \Phi) \sim \Pi_i \, (f_i - \Phi)^{e_i} \left(1 + \frac{\sigma^2}{4n^2\omega^2} \Sigma^2\right), \tag{52}$$

where f_i are the poles of the ratio ω/Σ and e_i are real numbers. The following toy model illustrates the simplest non-trivial scenario.

3.3   Toy model

Let us consider the simplest parametrization (40) that we have discussed at the end of §3.1 in a model where ω is in most regions sufficiently large, so that $\eta \gg \Phi$, and therefore $\omega \approx \eta/\Phi$. Thus, $\Sigma \approx \omega$. The EoI boundary is then given by $\Phi \approx 3\sigma^2/(2n)^2$ and it follows that

$$\mathcal{P}_c(\sigma, \omega, t) = C_0 \left(\frac{\sigma}{2n\eta}\right) \omega \, (\omega + 1)^{\frac{\gamma_1}{2}+1} (\omega + 2)^{-\gamma_1 - 1} \, \mathrm{e}^{-\gamma_1 t} \sim \sigma \, \omega^{1 - \frac{\gamma_1}{2}} \, \mathrm{e}^{-\gamma_1 t}, \tag{53}$$

and the corresponding volume ratio

$$r(\sigma, \omega) \sim \left(\frac{\omega^2}{\omega + 2}\right) \left[\frac{\omega + 1}{(\omega + 2)^2}\right]^{\gamma_1/2} \left[1 + \frac{\sigma^2}{4n^2} \left(\frac{\omega + 2}{\omega}\right)^2\right]^{1/2} \sim \omega^{1 - \frac{\gamma_1}{2}} \left(1 + \frac{\sigma^2}{4n^2}\right)^{1/2}. \tag{54}$$

For a typical value $\gamma_1 \sim 8$, (53)(54) predict $r \sim \omega^{-3}$, i.e. smaller values of ω are likelier than larger ones and, as indeed according to the Ansatz (40) ω decreases during the course of inflation, the most typical value within this model is $\omega_* = \eta/\Phi_{\mathrm{max}}$, where Φ_{max} is as before given by (41). A tendency towards greater ω requires a fine-tuning of parameters to orchestrate $\gamma_1 < 2$, at any rate in contradiction with the inequality $\gamma_1 > 3$ that must be satisfied for inflation to be eternal. Following (54) therefore most regions in the universe are occupied by $\omega(\Phi_{\mathrm{max}})$, that prevails as the most typical value of ω after inflation. In conclusion, $\omega(\Phi)$ is determined via (42) by the astrophysical determination of $\omega_{\mathrm{typical}}$ and a choice of potential that yields Φ_{max}.

Another toy model to the next order in (39) is

$$\omega = -1 + \frac{\eta_1}{\Phi} + \frac{\eta_2}{\Phi^2}, \tag{55}$$

where again we assume that ω is large compared to unity. (55) can be inverted, i.e.

$$\Phi \approx \frac{\eta_1}{2\omega} \left[1 + \left(1 + 4\frac{\eta_2}{\eta_1^2} \omega\right)^{1/2}\right], \tag{56}$$

or if η_1, η_2 are of comparable magnitude, then we have $\Phi \approx (\eta_2/\omega)^{1/2}$ and $\Sigma \approx 2\omega$. Therefore

$$\mathcal{P}_c(\sigma, \omega, t) \approx C_0 \left(\frac{\sigma}{4n} \right) \left(\frac{\omega}{\eta_2} \right)^{1/2 - \gamma_1/8} e^{-\gamma_1 t}, \tag{57}$$

and

$$r(\sigma, \omega) \approx \omega^{1 - \gamma_1/8} \left(1 + \frac{\sigma^2}{4n^2} \right)^{1/2}. \tag{58}$$

In (58) we find the same tendency as in (54) towards smaller values of ω. This tendency is less manifest in (58) and for a conservative value of $\gamma_1 = 8$ (58) appears to be independent of ω, due to the crude approximation $\eta_1 \sim \eta_2$ involved in the derivation of (57)(58). A more detailed numerical derivation of (58) shows that for arbitrary η_1, η_2, in fact one obtains $r \sim \omega^{1 - \gamma_1/\alpha}$, with $3.4 < \alpha < 3.9$, which is compatible with the conclusions of the simplest toy model (40).

3.4 Asymptotic regime

In the extreme case where $\Phi_0 \sim 0$ and $\Phi_{\max} \to \infty$, it is easy to see that any given Ansatz of the type

$$\omega(\Phi) = -1 + \frac{\eta_1}{\Phi} + \frac{\eta_2}{\Phi^2} + \ldots + \frac{\eta_M}{\Phi^M}, \tag{59}$$

i.e. a truncated version of (39), is dominated by the lowest and highest orders in the asymptotic regimes:

$$\omega(\Phi_0) \approx \frac{\eta_M}{\Phi_0^M} \tag{60}$$

and

$$\omega(\Phi_{\max}) \approx -1 + \frac{\eta_1}{\Phi_{\max}}. \tag{61}$$

If it is correct to assume that at the initial time the homogeneous bubble that undergoes inflation is in a string theory ground state, then $\omega(\Phi_0) \approx -1$ and $\eta_M \approx -\Phi_0^M$. From the astrophysical point of view, (61) represents the asymptotic behaviour near the EoI boundary, and is of interest as most regions of the universe are occupied by $\Phi = \Phi_{\max}$. Therefore, in the asymptotic regime any model of the type (59) is reduced to the toy model (40) that we have discussed in the previous section.

The relative values of the coefficients η_i cannot be determined directly from the hyper-extended inflation action and they are fixed by higher-loop estimates in string theory. If $\eta_2/\eta_1 \gg 1$, then the relative magnitude of η_2 with respect to $\Phi_{\max}$ determines whether (61) requires a higher-order correction by adding the term $\eta_2/\Phi_{\max}^2$ on the RHS. So far our calculations are based on the assumption that the η_i are well-behaved and therefore (61) can be considered a good approximation given these provisos.

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3.5 Spectrum of fluctuations

An ensemble of observers located on a homogeneous hypersurface (σ,Φ)=const observes quantum jumps of the fields due to the stochastic nature of inflation. On the one hand, there is the contribution of quantum fluctuations stretched beyond the horizon distance, that is effectively a stochastic force that acts on the classical solutions (30)-(32), such that

$$\dot{\sigma} = -2\Phi\left(\frac{V'}{V}\right) H + \frac{H^{3/2}}{2\pi}\,\zeta(t), \tag{62}$$

$$\dot{\Phi} = 2\left(\frac{\Sigma}{\omega}\right) H\Phi + \frac{H^{3/2}}{2\pi}\,\xi(t), \tag{63}$$

where ζ,ξ follow a Gaussian distribution, and $\langle\zeta(t_1)\zeta(t_2)\rangle = \langle\xi(t_1)\xi(t_2)\rangle = \delta(t_1 - t_2)$ and $\langle\zeta(t_1)\xi(t_2)\rangle = 0$. The second terms on the RHS of (62)(63) are random fluctuations of the fields that are superimposed on the slow-roll [classical] solutions over distances greater than H^{-1} (first terms on the RHS of (62)(63)). This is the so-called "coarse-grained" description of the fields. On the other hand, quantum jumps that take the fields to greater values are likelier than those that take them to smaller ones, because the physical volume occupied by the former is far greater. The volume ratio's dependence on the fields is rather steep in most cases, so fluctuations that end up in hypersurfaces of greater values are enhanced. This enhancement factor is approximately $\sim \mathcal{V}_B/\mathcal{V}_A$, where $\mathcal{V}_A$ is the volume of the hypersurface where the observers are located and $\mathcal{V}_B$ the volume of the hypersurface where the fields end up as the result of a fluctuation $(\delta\sigma,\delta\Phi)$. As a consequence of the combination of this factor and the stochastic nature of inflation, the typical quantum jumps $(\delta\sigma,\delta\Phi)$ observed by the average observer follow the distribution

$$d\mathcal{P}(\delta\sigma, \delta\Phi) \sim \frac{\mathcal{V}(\sigma + \delta\sigma, \Phi + \delta\Phi)}{\mathcal{V}(\sigma, \Phi)}\, d\mathcal{P}_0(\delta\sigma, \delta\Phi), \tag{64}$$

where $d\mathcal{P}_0$ is the Gaussian probability distribution of the stochastic field fluctuations,

$$d\mathcal{P}_0(\delta\sigma, \delta\Phi) = \frac{1}{(2\pi\Delta)^{1/2}}\,\exp\left[-\frac{(\delta\sigma)^2 + (\delta\Phi)^2}{2\Delta^2}\right] d\delta\sigma\, d\delta\Phi, \tag{65}$$

and the variance of the fields $\Delta = \langle\delta\sigma\rangle = \langle\delta\Phi\rangle = H/(2\pi)$. The distribution (64) is clearly non-Gaussian, as it is the product of a non-Gaussian distribution (the volume ratios) and a Gaussian one (the stochastic motion of the fields).

The stationary values of (64) yield the expectation values of the quantum jumps, $\langle\sigma\rangle$ and $\langle\Phi\rangle$, out of which we compute the spectrum of density fluctuations. The volume ratio $\mathcal{V}(\sigma + \delta\sigma, \Phi + \delta\Phi)/\mathcal{V}(\sigma, \Phi)$ has in general a complicated form, as can be seen from (48) and we simplify the calculation by examining the toy model (40) for a powerlaw potential. As we have seen in §3.3-4, (40) is a good approximation near the EoI boundary, which is the regime of interest for the spectrum of fluctuations. In this case, the volume ratio of the two

hypersurfaces becomes

$$\frac{\mathcal{V}(\sigma + \delta\sigma, \Phi + \delta\Phi)}{\mathcal{V}(\sigma, \Phi)} \approx \left[\frac{1 + (\sigma + \delta\sigma)^2/4n^2}{1 + \sigma^2/4n^2}\right]^{1/2} \left(1 + \frac{\delta\Phi}{\Phi}\right)^{\gamma_1/2}, \tag{66}$$

and therefore

$$\langle\delta\sigma\rangle \approx \frac{\sigma H^2}{16\pi^2 n^2}, \tag{67}$$

$$\langle\delta\Phi\rangle \approx \Phi. \tag{68}$$

In order to compute the spectrum of fluctuations we use the standard result for the adiabatic energy perturbations in a CDM-dominated universe [27, 22]

$$\left\langle\frac{\delta\rho}{\rho}\right\rangle = -\frac{6}{5}H\left[\dot\sigma\,\delta\sigma + 2\frac{(\omega + \Phi\omega')^2}{\omega}\frac{\dot\Phi}{\Phi}\delta\Phi\right]\left(\dot\sigma^2 + 2\frac{\omega}{\Sigma^2}\frac{\dot\Phi^2}{\Phi}\right)^{-1}, \tag{69}$$

where we have transformed the standard notation to the hyperextended inflation formalism. Taking into account the approximation $\Sigma \approx \omega$ in (40) and the results (67)(68) we get

$$\left\langle\frac{\delta\rho}{\rho}\right\rangle \approx \frac{\sigma^2}{20n^2}\frac{H^2}{(2\pi)^2\,\Phi}, \tag{70}$$

which is to be evaluated at $N \approx 65$ e-foldings after inflation. The dependence on σ can be eliminated via (35) and is dominated by a constant term dependent on the initial fields, i.e.

$$\left\langle\frac{\delta\rho}{\rho}\right\rangle \approx \frac{1}{20\pi^2}\left(\frac{\Phi_0}{\Phi}\right)H^2, \tag{71}$$

so essentially if we consider (as we have in the previous sections), that the predominant value of Φ in a physical volume of the universe of the horizon scale is Φ_{max} (so that $M_P(\Phi_{max}) \sim 10^{19}$ GeV) then the typical density contrast becomes $\langle\delta\rho/\rho\rangle \sim \Phi_0/\Phi_{max}$. Therefore if one is to adjust this to the astrophysical constraint $\langle\delta\rho/\rho\rangle \lesssim 10^{-4}$, then $\Phi_{max} \sim 10^{18}$ GeV is only compatible with a sufficient period of evolution, so that a much smaller initial value $\Phi_0 \sim 10^{13} - 10^{14}$ GeV can grow in several orders of magnitude to reach Φ_{max} at the end of inflation. This entails at the same time, within the model $\omega \approx \eta/\Phi$ a decrease of the same order of magnitude in the value of ω with respect to its initial value.

4 How typical is GR?

The notion of something being *typical*, in cosmology as in any other field, is defined within the context of an ensemble. In order to address the question of whether an object, quantity or phenomenon is typical or not we need to have a knowledge of the entire set or phase space that is accessible to us with the physics and initial conditions we set out with. In theories of hyperextended inflation, the larger ensemble that sets our standard of reference to define

what is typical is the universe as a whole.

In a quantum cosmological model that is governed by the hyperextended inflation action (1) the ensemble we need to consider is the physical space that results from the evolution after an arbitrary lapse of time. We know from the principle of stationarity that the properties of the inflating and non-inflating regions, volume ratios, etc, swiftly reach asymptotic values, and therefore we can safely investigate the likelihood of physical quantities regardless of the age of the universe. In fact, the principle of stationarity enables us to extend the evolution to the infinite past and to think of the notion of what is typical in an universe of arbitrary size. Everything is this model is ultimately determined by the choice of a potential; a choice that is made ad hoc or at best motivated by particle physics and indirect constraints from structure formation models. The potential determines the global structure of the universe, the ratio of inflating regions and non-inflating ones, it influences the resulting spectrum of fluctuations at the end of inflation, and it determines the distribution of values of the Planck mass as well as the coupling ω via the equations (30)-(32) that we have investigated in §3. In general, the choice of potential determines whether a given physical quantity is typical or not.

4.1 Likelihood, naturalness and typical quantities

Let us agree then on the convention that a physical quantity is *typical* when it is most probable within a universe that is governed by hyperextended inflation. In this Section we would like to address the question of whether GR is typical in this framework. GR is in essence reproduced by BD gravity in the limit of large ω, and therefore we ought to look at how typical this situation is. The regions that are still undergoing inflation are of no direct relevance to this discussion and we are interested in the values of the fields at the EoI boundary, after which M_P and ω remain essentially constant. Therefore we examine regions that have thermalized or are located at the neighbourhood of the EoI, to see if a typical region of this kind is compatible with GR.

In §3 we have seen that a potential with a finite value of Φ_{max} is the only likely framework to reproduce GR, and $\Phi_{\mathrm{max}} \to \infty$ entails that the likeliest value of ω is -1 as can be seen from (61). Theories that yield a finite Φ_{max} given by (41) are to a good approximation well described by the Ansatz (40) near the EoI boundary and lead to the likeliest value $\omega_* \approx \eta/\Phi_{\mathrm{max}}$, and η is determined by the initial conditions. In fact, as one can see from (54)(58), the probability distribution for ω is not extremely steep, it is typically a powerlaw and therefore, strictly speaking, a typical region of the universe has a value of Φ that is spread over a small range within the sharp wedge between the curves EoI and BoI (thick solid and dotted curves respectively) in Fig. 1. The values of Φ that fall within this neighbourhood $\lesssim \Phi_{\mathrm{max}} \sim 10^{18}$ GeV lead to values of ω that are, to a certain extent, also typical. An appropriate choice of these parameters such that ω_* satisfies the rather conservative constraint $\omega \gtrsim 500$ would therefore make GR a typical theory.

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The next question arises as to whether one should allow a greater freedom for a choice of potential, or whether it is physically *natural* to choose one, and to constrain the parameters conveniently so that one derives a result that tells us that GR is a typical theory. It could be argued that a potential ought to be picked out for its likelihood by the dynamics, because it *prevails* as a typical theory in a framework that allows all possible potentials, rather than by starting out with an ad hoc choice. In principle, one can undertake this step further and envisage a quantum cosmological model where all possible inflation potentials find a realization, such as in Fig. 3. The universe is then subdivided in an infinite number of regions v_i or *subuniverses* that are characterized by a given potential $V_i(\sigma)$. An arbitrary region $\mathcal{V}_i$ of the universe is therefore totally equivalent to the hyperextended theory that we have investigated. The universe as a whole however is not and it is not correctly described by the likelihood ratios (49) that we have computed for a powerlaw potential. To find the correct likelihood ratios we need to integrate the volumes of the homogeneous hypersurfaces $\mathcal{V}(\sigma, \Phi)$ over the entire space $\mathcal{M}$ of potentials, i.e. $\tilde{\mathcal{V}}(\sigma, \Phi) \equiv \int_{\mathcal{M}} dV \, \mathcal{V}(\sigma, \Phi)$. It is easy to see that quantities that may be typical within a subuniverse $\mathcal{V}_i$ may be not only not typical within the larger ensemble $\{\mathcal{V}_j\}$, but also probably highly unlikely.

It is quite apparent that investigating whether GR is typical is not without assumptions, some of which can be conflicting. We can classify the following two sets of inequivalent assumptions:

I Assume GR is typical, and therefore the possible inflaton potentials are restricted to the class $\Phi_{\max} < \infty$ and the free parameters are of the adequate order of magnitude so that $\omega_* \approx \eta/\Phi_{\max} \gtrsim 500$.

II Assume a certain inflaton potential(s), either from particle physics or reconstruction of the potential, and therefore we can conclude whether GR is typical or not depending on whether $\Phi_{\max} < \infty$ or $\Phi_{\max} \to \infty$ and the value of $\omega(\Phi_{\max})$.

Both approaches are clearly inequivalent and it can be argued that they can be motivated for different reasons. Assumption (I) is based on the so-called *principle of mediocrity* [28], which states briefly that the physical quantities observed in our neighbourhood of the universe are typical quantities, given that there is nothing especial about the region of the universe we inhabit with respect to other regions. Therefore our immediate neighbourhood cannot be singled out as a region that is untypical in any way, and GR has to be a perfectly typical theory. [5] In this context, the immediate implication is that the class of potentials that would make this possible is restricted by the conditions $\Phi_{\max} < \infty$ and $\omega_{\text{typical}} \gtrsim 500$. Therefore, the quantum cosmological scenario that we briefly summarized above and depicted in Fig. 3

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[5]It must be noted that, strictly speaking, the principle of mediocrity is not a good name for this notion, since *mediocrity* not only entails being average and unremarkable but, rather, below the average and underachieving. There is no reason to choose such negative connotations for a principle that is meant to denote what is likely, average and typical.

would be somewhat difficult to sustain under the assumption (I). Such a model would require that not all potentials that are possible are realized with equal probability but that those that satisfy the restrictions imposed by (I) are far more probable and prevail.

Assumption (II) involves coming up with a potential, that is motivated by observations or particle physics. For example, a potential that is a quasi-powerlaw with a kink on its slope that results in an open inflation theory is, it can be argued, well motivated by observations. The question of whether the potential chosen yields the appropriate volume ratios like those that we have computed in §3, in order to conclude that GR is a typical theory is a different matter. One needs to check this by computing the distribution of ω for each particular potential. In some cases GR may be a typical theory, in others it may not.

4.2 Is there a typical inflaton potential?

Following the analysis in §3, where we have based the notion of likelihood on the physical volume occupied by a given hypersurface, we can extend this notion to investigate how typical some potentials are with respect to others. Certainly, the potentials that yield $\Phi_{\max} \to \infty$ will be the likeliest ones, as they do by far occupy the largest physical volume (although the number of potentials that yield $\Phi_{\max} < \infty$ is far greater). In these theories, the fields can grow without limit during the course of inflation and occupy an arbitrarily large volume. Amongst this class of potentials, the relative likelihood of two potentials is difficult to quantify. Within the class of potentials that yields a finite $\Phi_{\max}$, there are powerlaw potentials ($n > 2$), exponential potentials or double-well potentials. For powerlaw potentials, we have seen from Fig. 1 that there are a fair amount of solutions, as an arbitrary $n > 2$ gives a finite $\Phi_{\max}$, whereas only $n = 1, 2$ yield $\Phi_{\max} \to \infty$.

If we adopt assumption (I) of §4.1, i.e. that GR is typical, then we conclude that the powerlaw potentials that have the smallest values of λ are favoured and are the likeliest, as we have from (41) that typically $\Phi_{\max} \sim \lambda^{-1/(n-2)}$. Therefore, a small value of λ such as (14) would be perfectly consistent with GR being a typical theory. According to this argument, hence even though particle physics does not provide a mechanism to discriminate values of λ, and all possible values of λ are feasible with equal probability, only the smallest ones turn out to be naturally the likeliest ones because they yield the largest $\Phi_{\max}$, and thus this creates an ideal scenario to accept that GR is a typical theory. Even though GR is indeed a typical theory within the ensemble of regions governed by powerlaw potentials, or at any rate not an unlikely one (using a more conservative viewpoint), GR is not a typical theory within the larger ensemble of all possible potentials, as these are donimated by $\Phi_{\max} \to \infty$ regions. These include a number of exotic potentials, such as logarithmic, etc. On the other hand, if we were to exclude from all possible realizations those potentials that are remotely exotic, and one just allowed a physical realization to a more conventional class of potentials, say e.g. powerlaw, exponential and double-well potentials, then within this restricted subclass $\Phi_{\max} < \infty$ is a predominant solution and we are back to the assertion that GR is a typical

theory.

It could be said that this is a rather contrived way of looking for a typical potential or for discarding potentials that modify the notion of what is typical in a way that we do not want. On the other hand, if we abandon the question of looking for a typical potential that leads to GR as the likeliest theory, and we allowed all possible potentials to find a realization, as we described in the previous section, we find that GR is an extremely unlikely and untypical theory. However one could argue that this might not be all that discouraging. From the anthropic point of view, this could be perfectly acceptable as an untypical configuration on which the very existence of human life and its relationship to the physical world relies. This viewpoint belongs to the class of assumptions (II). From what we have seen, it is apparent that adopting either (I) or (II) can lead to completely different notion of how typical GR is.

5   Conclusions

We have investigated some aspects of extended and hyperextended inflation that are related to the question of how typical GR is. Two important results are the derivation of the conservation laws (8)(35), which give us the classical trajectories of the fields between the boundaries BoI and EoI. The quantum fluctuations of the fields, due to the stochastic nature of inflation, are superimposed on the classical trajectories, and as we have discussed in §3, these contribute to a significant extent to drive the fields to larger values. It is fundamental to take this effect into account in computing the likelihood of a observer being in a region (σ,Φ)=const, that is proportional to the physical volume $\mathcal{V}(\sigma,\Phi)$ of the corresponding homogeneous hypersurface.

We have also divided inflaton potentials in two classes:

A   The BoI and EoI curves intersect, therefore there is a maximum value of the BD field $\Phi_{\mathrm{max}} < \infty$ for which inflation can still take place,

B   BoI and EoI do not intersect, and therefore $\Phi_{\mathrm{max}} \to \infty$.

The first class of potentials is of interest and has been employed in most calculations in §3, whereas the second class is problematic in that the fields grow to arbitrarily large values and the most typical scenario is one of $\omega = -1$ and a departure from this value takes place with negligible probability. These so-called "run-away" solutions are discussed in §2.1. Fortunately, most powerlaw potentials ($n > 2$) belong to class A, and our analysis of hyperextended inflation in §3 has been applied to these. It is straightforward to derive the same results for arbitrary potentials, of either class A or B, as the main equations presented are valid for generic potentials, provided the slow-roll approximation applies.

In §3 we have investigated hyperextended inflation theories making an Ansatz for $\omega(\Phi)$ based on the principle of least coupling. The coefficients η_i in (39) that determine ω are given by loop corrections of the lowest-order effective action in string theory, and these are not easy to calculate. However, for the toy models that we have considered in §3.3–4, one can constrain the coefficients in terms of the initial conditions, and in general, in terms of the approximate behaviour in the vicinity of EoI as described in §3.4. Other constraints on η_i can be derived for example from the bubble size spectrum. As is well known, extended inflation suffered from the so-called *big-bubble problem* (see e.g. last reference in [17]). Hyperextended inflation enables us to adjust the parametrization of ω and fit η_i as best as possible to avoid this problem in particular.

In §3 we have seen that powerlaw potentials of class A are to a large extent consistent with GR being a typical theory whereas for those in class B, GR is very untypical. In §4 we have discussed the conceptual framework of a quantum cosmological framework that is governed by hyperextended inflation. A scenario that allows a realization of all possible scalar potentials results in an ensemble of subuniverses $\mathcal{V}_i$ each one of which is described by the distributions computed in §3. In this senario GR is not a typical theory as most of the volume is dominated by regions governed by potentials of class B. In a more restricted scenario that only allows a realization for potentials of class A (that also happen to be the less exotic potentials and physically more realistic), it is consistent that GR is a typical theory.

In conclusion, it must be said that the notion of how typical GR is, as presented in the title of this paper, is a question that escapes a straightforward answer. As we have discussed in §4, the two different assumptions (I) and (II) can lead us to reach a different conclusions, and the uncertainty derives from our lack of knowledge of the inflaton potential. However many things can be said about whether GR is typical or untypical depending on the potential chosen. The scenario that allows a realization of all possible potentials $\Phi_{\max} < \infty$ is a particularly promising framework, that results in GR being a typical theory.

Appendix. Volume ratios in extended inflation

In this appendix we work out the details of the derivation of (20). From (16)(17) we have

$$J_\sigma \approx -2\Phi \left(\frac{V'}{V} \right) \mathcal{P}_c, \tag{72}$$

$$J_\Phi \approx 2 \left(\frac{\Phi}{\omega} \right) \mathcal{P}_c, \tag{73}$$

where $\mathcal{P}_c$ is given by (18). As was shown in [22], the comoving probability $\mathcal{P}_c$ is strongly dependent of the choice of time variable. It is desirable thus to use a measure of the likelihood

of the values of the fields that is independent of the choice of time parameter. The principle of stationarity tells us that, given that a sufficiently long period of inflation has elapsed, the volume of a hypersurface $\mathcal{V}(\sigma, \Phi)$ becomes a constant fraction of the total volume of the universe. Of course the total volume of the universe $\mathcal{V}_T$ grows indefinitely and the volumes of homogeneous hypersurfaces are also unbounded, but the volume ratio of two homogeneous hypersurfaces or that of a homogeneous hypersurface with respect to the total volume of the universe remains finite. Throughout this article we have arbitrarily chosen that the measure of likelihood of the fields (σ, Φ) is given by the ratio of the volume of the hypersurface (σ, Φ)=const to the volume occupied by thermalized regions:

$$r = \frac{\mathcal{V}(\sigma, \Phi)}{\mathcal{V}_*}, \tag{74}$$

where

$$\mathcal{V}(\sigma, \Phi) = \mathcal{V}_0 \left| \int_0^{t_c} \mathrm{d}t \, \mathrm{e}^{3t} \left(\vec{J} \cdot \hat{l} \right) \right|, \tag{75}$$

$$\mathcal{V}_* = \mathcal{V}_0 \left| \int_0^{t_c} \mathrm{d}t \, \mathrm{e}^{3t} \int_{\mathrm{EoI}} \mathrm{d}l \, \left(\vec{J} \cdot \hat{n} \right) \right|, \tag{76}$$

where $\mathcal{V}_0$ is an arbitrary volume of an initial homogeneous hypersurface, $\hat{l}$ is a tangent vector to the curve (8) at (σ, Φ), $\hat{n}$ is a normal vector to the EoI boundary (4). The parameter t_c is an arbitrarily large *cut-off time* that regularizes the volumes (75)(76). The volume ratio (74) is independent of t_c as shown in [25, 21] and therefore we take the limit $t_c \to \infty$.

After some algebra it is easy to show that in the case of a powelaw potential

$$\mathcal{V}(\sigma, \Phi)_{\mathrm{regularized}} = 2\mathcal{V}_0 \left\{ \frac{\exp\left[(3 - \gamma_1) \, t_c\right] - 1}{3 - \gamma_1} \right\} \Phi^{\frac{\omega}{2}\gamma_1} \left(1 + \frac{\sigma^2}{4n^2 \, \omega^2} \right)^{1/2}, \tag{77}$$

$$\mathcal{V}^*_{\mathrm{regularized}} = \mathcal{V}_0 \left\{ \frac{\exp\left[(3 - \gamma_1) \, t_c\right] - 1}{3 - \gamma_1} \right\} \frac{4}{\omega \gamma_1} \Phi^{\frac{\omega}{2}\gamma_1 + 1}_{\mathrm{max}}. \tag{78}$$

Therefore the volume ratio is

$$r(\sigma, \Phi) = \frac{\omega}{2} \gamma_1 \, \Phi^{-\frac{\omega}{2}\gamma_1 - 1}_{\mathrm{max}} \left(1 + \frac{\sigma^2}{4n^2 \, \omega^2} \right)^{1/2} \Phi^{\frac{\omega}{2}\gamma_1} \tag{79}$$

which is the result that has been used in (20) as a measure of the likelihood of a given configuration (σ, Φ). The presence of Φ_{max} arises from the integral over the EoI boundary in (76). This quantity, like γ_1 is model-dependent. For powerlaw potentials $n \leq 2$ we have seen however in §2 that $\Phi_{\mathrm{max}} \to \infty$. According to (79) in that limit only the fields at infinity would yield a non-vanishing value of r, but we rule out that possibility as it would make our region of the universe an extraordinarily unlikely one.

References

[1] Linde A.D., 1990, *Particle Physics and Inflationary Cosmology*, Harwood, Chur, Switzerland

[2] Got J.R. III, 1982, Nature 295, 304; Sasaki, M., Tanaka T., Yamamoto K. and Yokoyama J., 1993, Phys. Lett. B317, 510; Bucher M., Goldhaber A.S. and Turok N., 1995, Phys. Rev. D52, 3314; Bucher M., Goldhaber A.S. and Turok N., 1995, Nucl. Phys. Proc. Suppl. 43, 173; Yamamoto K., Tanaka T. and Sasaki M., 1995, Phys. Rev. D51, 2968; Bucher M., Turok N., 1995, Phys. Rev. D52, 5538; Barrow J.D., Liddle A.R., 1997, gr-qc/9705048; Bucher M., Cohn J.D., 1997, Phys. Rev. D55, 7461; Vilenkin A., 1997, Phys. Rev. D56, 3238

[3] Linde A.D., 1992, Nucl. Phys. B372, 421; Kamionkowski M., Toumbas N., 1996, Phys. Rev. Lett. 77, 587; White M., Scott D., 1996, Astrophys. J. 459, 415

[4] Dekel A., 1994, Ann. Rev. Astron. Astrophys. 32, 371-418; Dekel A., 1996, 'Measuring Omega' in *Critical Dialogues in Cosmology*, proceedings of the Princeton 250th Anniversary Conference, ed. Turok N., World Scientific; Dekel A., 1997, 'Cosmological implications of large-scale flows' and references therein, astro-ph/9705033, to appear in proceedings of the 3rd ESO-VLT Workshop on *Galaxy Scaling Relations: Origins, Evolution and Applications*, ed. da Costa L., Springer; Coles P., Ellis G.F.R., 1997, *"Is the Universe Open or Closed?"*, Cambridge University Press, Cambridge

[5] It is currently expected that data from the *Planck/First Surveyor* will pin down cosmological parameters to an accuracy of approximately 10%, when the project will be carried out in 2006; see e.g. Frieman J., 1997, in *Fundamental Physics at the Birth of the Universe II*, Rome; Kamionkowski M., *ibid.*

[6] Linde A.D., 1982, Phys. Lett. 108B, 389; Albrecht A., Steinhardt P.J., 1982, Phys. Rev. Lett. 48, 1220; Linde A.D., 1983, Phys. Lett. 129B, 177; Bardeen J.M., Steinhardt P.J. and Turner M.S., 1983, Phys. Rev. D28, 679; see also Kolb E.W., Turner M.S., 1990, *The Early Universe*, Addison-Wesley, Reading MA

[7] McDonald J., 1993, Phys. Rev. D48, 2462

[8] Hartle J.B., Hawking S.W., 1983, Phys. Rev. D28, 2960; Hartle J.B., 1991, "The Quantum Mechanics of Cosmology" in *Quantum Cosmology and Baby Universes*, Proceedings of the 1989 Jerusalem Winter School, eds. Coleman S., Hartle H.B., Piran T., and Weinberg S., World Scientific, Singapore

[9] Vilenkin A., 1995, Phys. Rev. Lett. 74, 846; Garcia-Bellido J., Linde A.D., 1995, Phys. Rev. D50, 730

[10] Starobinsky A., 1986, *Current Topics in Field Theory, Quantum Gravity and Strings*, Eds. de Vega H.J. and Sanchez N., Lecture Notes in Physics 206, Springer, Heidelberg

[11] Linde A.D., Mezhlumian A., 1993, Phys. Lett. B307, 25; Linde A.D., Linde D.A. and Mezhlumian A., 1994, Phys. Rev. D49, 1783; These references investigate the distribution $P_p(\sigma, t)$ and conclude that it swiftly approaches a stationary regime, which is strongly dependent on the choice of t.

[12] Brans C., Dicke R.H., 1961, Phys. Rev. 124, 925; Dicke R.H., 1961, Phys. Rev. 125, 2163; Brans C.H., 1961, Phys. Rev. 125, 2194

[13] Sciama D.W., 1953, MNRAS 113, 34

[14] Damour T., Vilenkin A., 1996, Phys. Rev. D53, 2981

[15] La D., Steinhardt P.J., 1989, Phys. Rev. Lett. 62, 376; La D., Steinhardt P.J., Bertschinger E., 1989, Phys. Lett. B231, 231; Mathiazhagan C., Johri V.B., 1984, Class. Quantum Grav. 1, L29

[16] Guth A.H., 1981, Phys. Rev. D23, 347

[17] Weinberg E., 1989, Phys. Rev. D40, 3950; La D., Steinhardt P.J. and Bertschinger E., 1989, Phys. Rev. Lett. B 231, 231; Liddle A.R., Wands D., 1991, MNRAS 253, 637; a discussion on the disagreement with the constraint $\omega \leq 25$ can be found in Goldwirth D.S. and Zaglauer H.W., 1991, Phys. Rev. Lett. 67, 3639, that supports the initial *extended inflation* model without modification.

[18] Reasenberg R.D. *et al.* , 1979, Astrophys. J. 234, L219; de Rujula A., 1986, Phys. Lett. B180, 213

[19] Barrow J.D., Maeda K., 1990, Nucl. Phys. B341, 294; Liddle A.R., Wands D., 1992, Phys. Rev. D45, 2665; Barrow J.D., Mimoso J.P., 1994, Phys. Rev. D50, 3746; Barrow J.D. Parsons P., 1997, Phys. Rev. D54, 1906

[20] Steinhardt P.J., Accetta F.S., 1990, Phys. Rev. Lett. 64, 2740

[21] Susperregi M.P.K., 1997, Phys. Rev. D55, 560

[22] Garcia-Bellido J., Linde A., Linde D., 1994, Phys. Rev. D50, 730

[23] Starobinsky A.A., Yokoyama J., 1994, Proc. 4th Workshop on General Relativity and Gravitation, Kyoto, Japan (also astro-ph/9502002)

[24] Damour T., Polyakov A.M., 1994, Nucl. Phys. B423, 532; Damour T., Polyakov A.M., 1994, Gen. Rel. Grav. 26, 1171

[25] Vilenkin A., 1995, Phys. Rev. D52, 3365

[26] Linde A., Mezhlumian A., 1996, Phys. Rev. D53, 4267

[27] Lyth D.H., 1985, Phys. Rev. D31, 1792

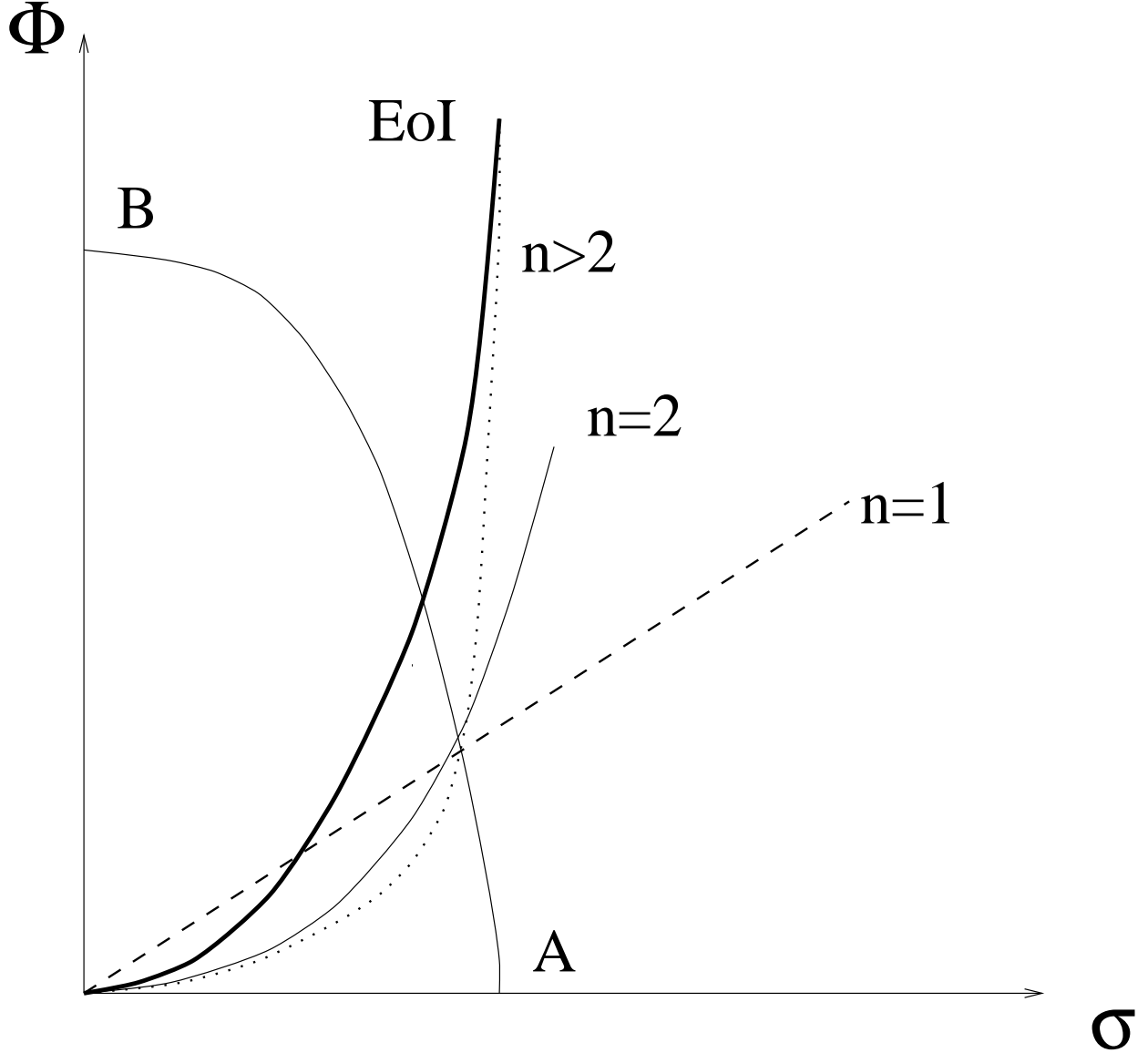[28] Vilenkin A., 1995, Phys. Rev. Lett. 74, 846

Figure 1: Classical evolution of the fields for a powerlaw potential. The solid curve AB represents the parabola (4). Given some initial conditions (σ_0, Φ_0) on AB, the fields roll along this curve in the direction $A \to B$, and quantum jumps take the fields from one classical trajectory to another. The thick solid line represents the EoI boundary. The BoI boundary is represented by: the dashed line for $n = 1$, solid line for $n = 2$ and dotted line for $n > 2$. EoI and BoI intersect at $\Phi_{\max}$ for $n > 2$.
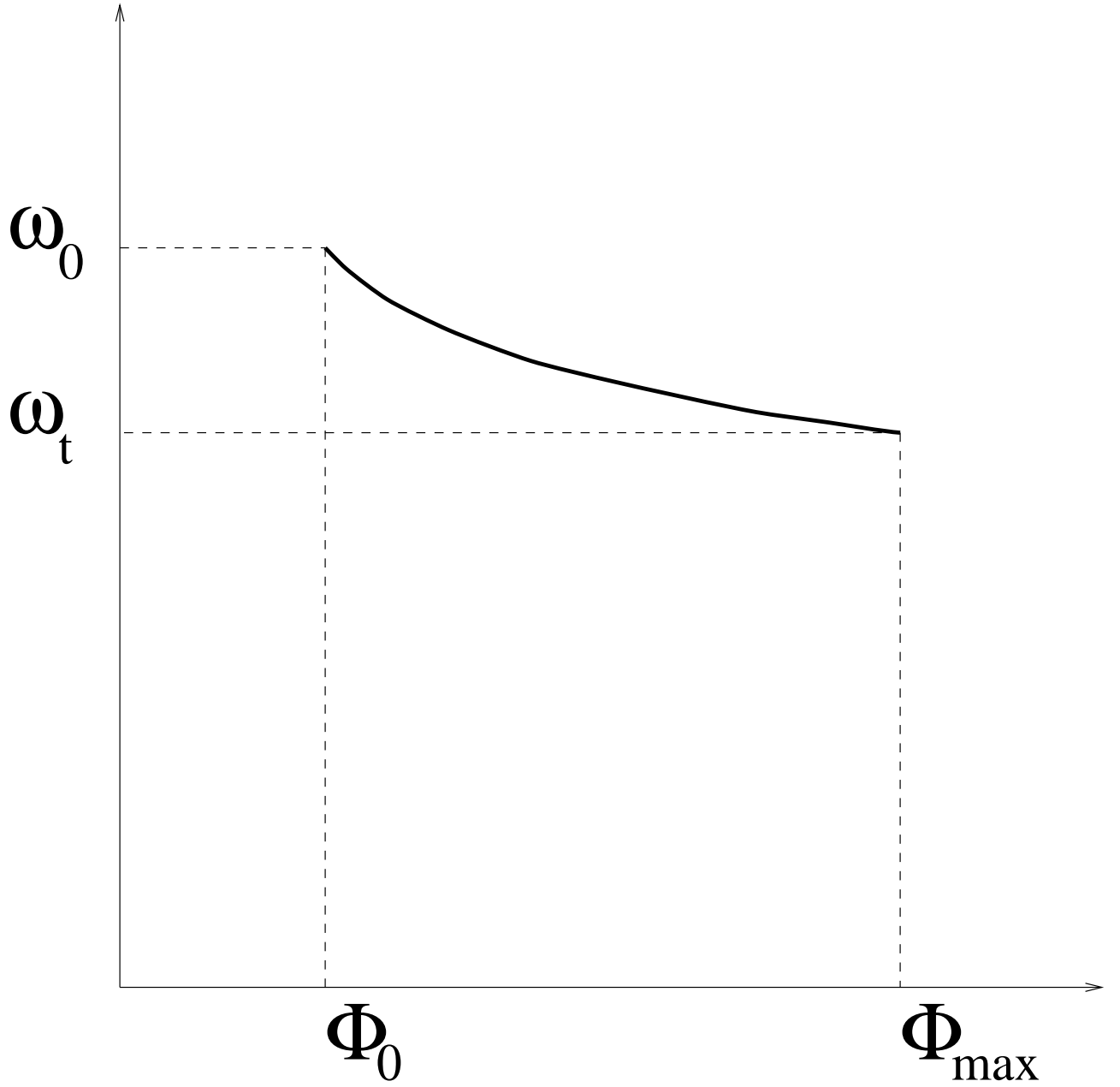
Figure 2: Evolution of ω for the toy model $\omega \sim \eta/\Phi$. The value of ω decreases during the course of inflation and becomes constant after crossing the EoI boundary. The lowest value of ω possible corresponds to $\Phi_{\max}$, the highest value of Φ along the EoI for which inflation still occurs. This is at the same time the likeliest or most *typical* value of ω.
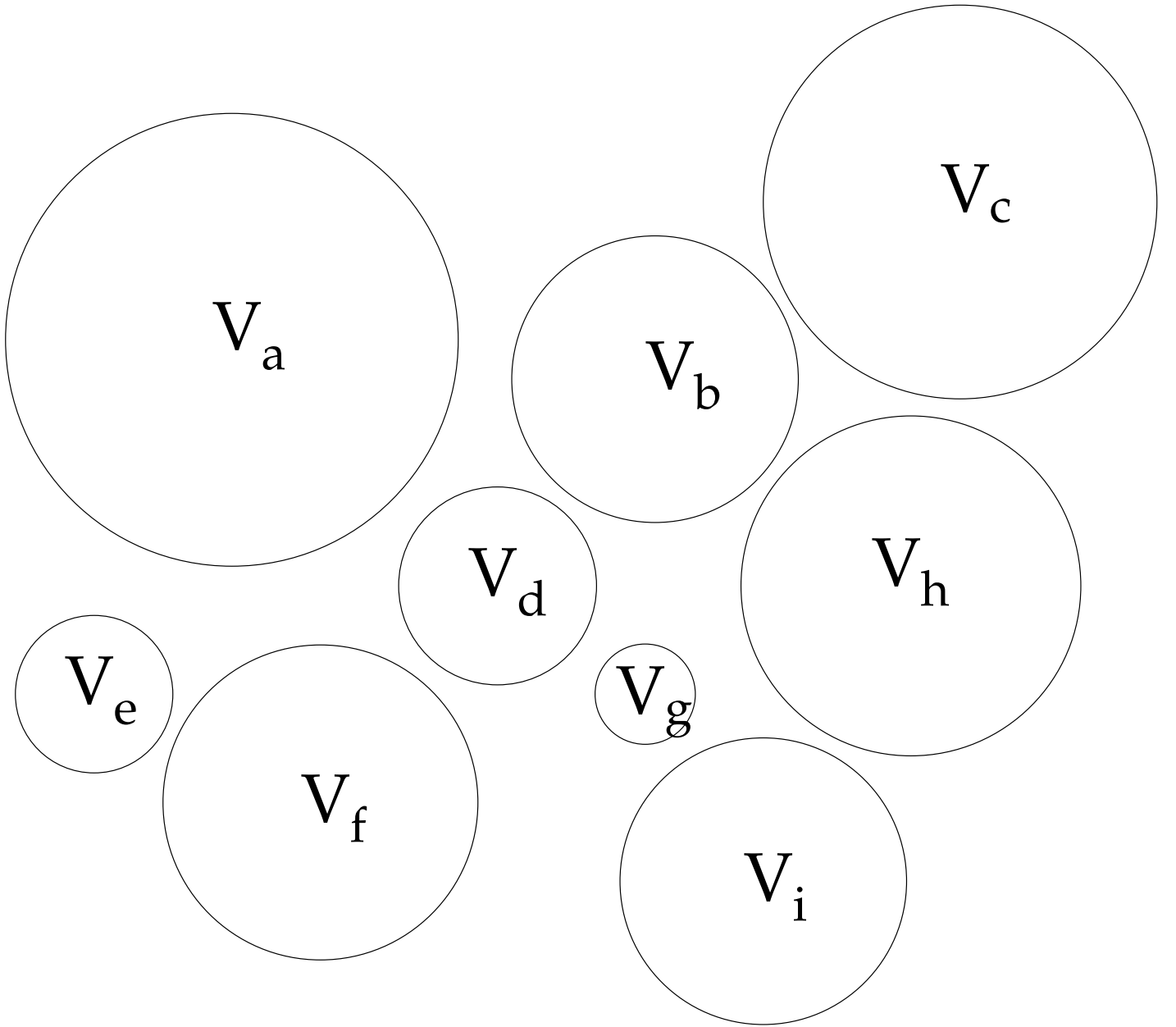
Figure 3: The hyperextended inflationary universe. An arbitrary region $\mathcal{V}_i$ is governed by a potential $V_i(\sigma)$ via the hyperextended inflation action (1). All possible potentials find a realization and are equally probable. The probability distributions of the fields and the volume ratios of homogeneous hypersurfaces within $\mathcal{V}_i$ are given by the equations of §3. Some of the $\mathcal{V}_i$ occupy a larger fraction of the entire volume of the universe than others, depending on the relative magnitude of $\Phi_{\max}^{(i)}$. An observer is naturally typically located in a region $\Phi_{\max} \to \infty$ as these regions span by far the largest fraction of the volume.